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# ELASTIC EQUILIBRIUM OF A MEDIUM CONTAINING A FINITE NUMBER OF ARBITRARILY ORIENTED SPHEROIDAL INCLUSIONS

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**Abstract**—An unbounded domain containing a certain number of spheroidal inclusions proves to be a suitable model for a wide class of heterogeneous materials. Elastic equilibrium of this model structure, in the case of aligned spheroids was studied, before by Kushch, who has obtained an accurate solution in series using the multiple expansion technique. The present paper expands this approach on the case of the arbitrarily oriented spheroids. The new formulae are obtained providing the re-expansion of the vectorial partial solutions of Lame's equation, due to the rotation of the coordinate system. The using of these formulae enables one to reduce a primary boundary-value problem to an infinite set of linear algebraic equations. Some numerical results are presented which demonstrate how the interfacial stress concentrations, caused by an interaction among the particles, vary with their relative position and orientation. © 1998 Elsevier Science Ltd.

## 1. INTRODUCTION

Under the theoretical description of elastic behaviour of a particle reinforced composite a complexity of the geometrical model used should agree with the values of structure parameters, primarily, with a volume content of disperse phase and a ratio of phase moduli. So, for the dilute weakly-heterogeneous composites the so-called "one-particle" models, consisting of a single ellipsoidal inclusion embedded into an infinite matrix, give a reasonable approximation of both the stress field and the effective elastic moduli of composite. There is a variety of theories based on this model. They use, as a rule, the well-known Eshelby's (1959) solution for a single ellipsoidal inclusion and differ mainly by the closure hypothesis: e.g. the self-consistent scheme by Hill (1965), the effective field theory by Levin (1977), the Mori–Tanaka's theory by Tandon and Weng (1986), etc.

However, with a volume content of disperse phase increased all the above theories become more and more inaccurate. The reason here is that in the high-filled composites the interactions between the neighbouring particles influence greatly their overall response. The extreme case is a strongly heterogeneous material with the near-to-dense packing of disperse phase particles which is characterized by the very high microstress concentrations. To predict the overall properties of such a composite accurately the interaction effects mentioned should be taken into account to a maximum possible extent. The natural way to make it is to consider a many-particle model which is an infinite matrix containing a certain number of particles arranged in an ordered or disordered array. The advantage of this model is an ability to approach a structure of actual composite. In particular, the spatial distribution rule of disperse phase particles, their size distribution, presence of several kinds of filler etc., can be taken into account properly within this model. At the same time, a more complex (although realistic) model requires a more complex mathematics to analyse it accurately. The lack of appropriate mathematical techniques convenient for use is, probably, the main reason why a number of publications where the many-particle models are studied is rather limited.

For the composites, filled by spherical particles, the methods of analysis are relatively well developed now. The elasticity problems for a space containing a finite number (two or more) or spherical cavities or particles were considered by Sternberg and Sadowsky (1952), Golovchan (1974), Tsuchida *et al.* (1976), Chen and Acrivos (1978), Rodin and Hwang (1993), Golovchan *et al.* (1993) and others. The accurate solutions for a periodic structure model of composite which is a matrix containing a three-dimensional periodic array (lattice) of identical spherical inclusions were obtained by Nunan and Keller (1984), Kushch (1985), Sangani and Lu (1987), Kushch (1987). The more general periodic models with an elementary structure cell containing a certain number of arbitrary placed non-touching inclusions were considered by Golovchan *et al.* (1993) and Sangani and Mo (1996).

The very attractive idea is to extend the approaches already developed on the composites with non-spherical (namely, ellipsoidal) particle's shape. This generalization seems important enough because it extends their applicability area on a wide class of heterogeneous materials including the composites reinforced by short fibres or platelets, materials with needle-like or penny-shaped cracks, metals and ceramics modified by phase transformation, etc. However, in a case of non-spherical inclusions the corresponding model boundary-value problems become much more difficult for analysis and only a few papers are available where the interactions between the ellipsoidal particles were studied. So, in the proposed "double-inclusion" model, by Hori and Nemat-Nasser (1993), the average field quantities are estimated approximately with the aid of Eshelby's (1959) solution and the theorem generalising the Mori–Tanaka theory. A more rigorous approach to the problem for a medium containing two ellipsoidal inhomogeneities was suggested by Moskovidis and Mura (1975). They made use of the representation of the transformation strains within each domain by a polynomial in Cartesian coordinates. The decomposition of the strain field around the inhomogeneities into a Taylor series reduces the differential problem to an algebraic set of equations. It is rather difficult to estimate the computational effectiveness of the method because the numerical results are presented for the widely separated inclusions only. Averaged elastic properties of a composite containing aligned periodically distributed ellipsoidal inclusions were estimated by Iwakuma and Nemat-Nasser (1983) under the assumption of uniform strain and stress fields inside the inclusions.

The problem on elastic equilibrium of a medium containing a finite number of aligned spheroidal inclusions has been solved accurately by Kushch (1996a). The essence of method used is representation of the displacement vector in a multiple-connected domain (matrix) as a sum of general solutions for the corresponding single-connected domains having a form of space with a single spheroidal hole. Each term of this sum is expanded, in turn, into a series on the partial vectorial solutions of Lame's equation in a spheroidal basis. In order to satisfy exactly all the interface boundary conditions the re-expansion formulae (addition theorems) are used. As a result, the primary boundary-value problem is reduced to an infinite set of linear algebraic equations. This approach proves to be efficient from a computational standpoint and provides an accurate and detailed analysis of the stress fields in each phase of many-particle model. The natural generalization of this model is a periodic composite with a structure cell containing a finite number of aligned spheroidal inclusions. The stressed state and the effective elastic moduli tensor of this type composite have been determined accurately by Kushch (1996b). The model mentioned has a number of parameters and provides generating the quasi-random structure close to the actual disordered composite.

The matter is, however, that in majority of composites with prolate or oblate particles the lasts are oriented randomly rather than unidirectionally. The adequate geometrical model here is a medium or a cell of a periodic structure containing a finite number of arbitrarily placed and arbitrarily oriented spheroidal inclusions because it can take into account not only the spatial distribution rule of disperse phase particles, but also the orientation statistics of microstructure of composite. Below the stressed state of this model structure due to the arbitrary uniform external loads applied is evaluated accurately. As before, the only geometrical restriction, namely, non-intersecting and non-touching of any two or more inclusions is imposed. The method used here is the direct generalization of the technique developed by Kushch (1996a). In order to satisfy exactly the interface conditions on the non-aligned spheroidal surfaces the new re-expansion formulae are derived.

## 2. STATEMENT OF THE PROBLEM

Let us consider a homogeneous isotropic elastic medium (matrix), containing $N$ arbitrarily oriented spheroidal inclusions. Their centres are located in the points $O_q$, $q = 1, 2, \ldots, N$. To describe a geometry of the problem we introduce for each particle a local Cartesian coordinate system $O_q x_q y_q z_q$ so that its origin lies in the point $O_q$ and the $O_q z_q$ axis coincides with the rotation axis of $q$th spheroid. We also define the spheroidal coordinate $(f_q, \xi_q, \eta_q, \phi_q)$ related with the Cartesian ones by the formula (A5) of Appendix A so that at the $q$th interface we have $\xi_q = \xi_q^{(0)}$. The semiaxes of $q$th spheroid are $p_x = p_y = f_q \bar{\xi}_q^{(0)}$, $p_z = f_q \xi_q^{(0)}$. Aspect ratio $\varepsilon_q = p_z/p_x = \xi_q^{(0)}/\bar{\xi}_q^{(0)}$ is the shape factor of spheroid, for a sphere $\varepsilon_q = 1$. We denote also as $\mathbf{r}_q$ the radius-vector of $q$th local coordinate basis. Relation between the components of the radius-vectors of two different local coordinate systems is given by the simple formula

$$\mathbf{r}_p = \mathbf{R}_{pq} + O_{pq} \cdot \mathbf{r}_q \tag{1}$$

where $\mathbf{R}_{pq}$ is the translation vector connecting the points $O_p$ and $O_q$ and $O_{pq}$ is an orthogonal rotation matrix with $\det O_{pq} = 1$. Formula (1) is nothing, but a general linear orthogonal transformation. It is common knowledge that a transformation of this kind can be decomposed into a sum of parallel transfer (translation) and rotation with respect to stationary origin of a coordinate system. A pair $(\mathbf{R}_{pq}, O_{pq})$ determines uniquely the reference position and orientation of the particles with indices $p$ and $q$. Finally, we introduce also a global Cartesian coordinate system $Oxyz$. Because there are no restrictions on a position of this system we choose it coinciding with the local basis of 1st spheroid, hence, $\mathbf{r} = \mathbf{r}_1$.

We suppose that the stressed state of a composite medium is induced by the constant strain tensor $\hat{E} = \{E_{ij}\}$ applied at infinity. The displacement vector $\mathbf{u}$ in each phase satisfies the Lame's equation

$$\frac{2(1-\nu)}{1-2\nu}\nabla(\nabla\cdot\mathbf{u}) - \nabla\times\nabla\times\mathbf{u} = 0 \tag{2}$$

where $\nu$ is the Poisson's ratio; $\mathbf{u} = \mathbf{u}^{(0)}$, $\nu = \nu_0$ in the matrix, $\mathbf{u} = \mathbf{u}^{(q)}$, $\nu = \nu_q$ in the $q$th particle. On interfaces the continuity conditions of displacement vector $\mathbf{u}$ and normal stress vector

$$\mathbf{T}_\xi = \sigma_\xi \mathbf{e}_\xi + \tau_{\xi\eta}\mathbf{e}_\eta + \tau_{\xi\phi}\mathbf{e}_\phi = 2\mu\left[\frac{\nu}{1-2\nu}\mathbf{e}_\xi \nabla\cdot\mathbf{u} + \frac{\xi\eta}{f}\frac{\partial}{\partial\xi}\mathbf{u} + \frac{1}{2}\mathbf{e}_\xi \times\nabla\times\mathbf{u}\right] \tag{3}$$

are prescribed:

$$[\mathbf{u}^{(0)} - \mathbf{u}^{(q)}]_{\xi_q = \xi_q^{(0)}} = 0; \quad [\mathbf{T}_{\xi_q}(\mathbf{u}^{(0)}) - \mathbf{T}_{\xi_q}(\mathbf{u}^{(q)})]_{\xi_q = \xi_q^{(0)}} = 0, \quad q = 1, 2, \ldots, N. \tag{4}$$

In the special case of $O_{pq} = I$ $(p, q = 1, 2, \ldots, N)$, where $I$ is the unit matrix (no rotations) we have exactly the problem considered by Kushch (1996a). We will follow the method and notations described in detail in the paper cited. To avoid duplicating, only the main idea of the solving procedure will be reproduced below.

## 3. METHOD OF SOLUTION

According to the generalized superposition principle we represent the displacement vector in continuous phase (matrix) $\mathbf{u}^{(0)}$ as a sum of the linear far field and the disturbance fields produced by each separate cavity or inclusion:

$$\mathbf{u}^{(0)} = \hat{E}\cdot\mathbf{r}+\sum_{p=1}^{N}\mathbf{U}^{(p)}. \tag{5}$$

Now we expand $\mathbf{U}^{(p)}$ into a series on the vectorial partial solutions of Lame's equation (Appendix A). Because at $\|\mathbf{r}\| \to \infty$ $\mathbf{U}^{(p)}$ tends to zero, this series development contains the external (singular) solutions (A1) only:

$$\mathbf{U}^{(p)} = \sum_{i=1}^{3}\sum_{t=0}^{\infty}\sum_{s=-t}^{t} A_{ts}^{(i)(p)}\mathbf{S}_{ts}^{(i)}(\mathbf{r}_p,f_p) \tag{6}$$

where $A_{ts}^{(i)(p)}$ are the unknown series coefficients are to be found from the boundary conditions (4). On the contrary, a series expansion of the displacement vector $\mathbf{u}^{(q)}$, finite and continuous in all the points of $q$th inclusion, contains the internal (regular) solutions $\mathbf{s}_{ts}^{(i)}$ (A2) only:

$$\mathbf{u}^{(q)} = \sum_{i=1}^{3}\sum_{t=0}^{\infty}\sum_{s=-t}^{t} D_{ts}^{(i)(q)}\mathbf{s}_{ts}^{(i)}(\mathbf{r}_q,f_q) \tag{7}$$

where $D_{ts}^{(i)(q)}$ are the arbitrary constants.

Because $\mathbf{S}_{ts}^{(i)}$ (A1) and $\mathbf{s}_{ts}^{(i)}$ (A2) are the partial solutions of Lame's equation, the displacement vector in a matrix (5) and in the inclusions (7) satisfies eqn (2) exactly. To execute the interfacial conditions (4) the unknowns $A_{ts}^{(i)(q)}$ and $D_{ts}^{(i)(q)}$ should be determined properly. To obtain the resolving set of linear algebraic equations for their evaluation we should make the following. First, with the aid of the addition theorems we express $\mathbf{u}^{(0)}$ in the variables of $q$th local coordinate basis. Then, by substituting the transformed expression of $\mathbf{u}^{(0)}$ together with $\mathbf{u}^{(q)}$, written already in this local basis, into eqns (4) we obtain $2N$ functional equalities. Finally, we decompose these vectorial equalities onto scalar ones and make use the orthogonality property of the set of scalar spherical harmonics $\chi_t^s(\eta_q, \phi_q)$ on the surface $\xi_q = \xi_q^{(0)}$ to obtain an infinite set of linear algebraic equations with unknowns $A_{ts}^{(i)(q)}$ and $D_{ts}^{(i)(q)}$. An analysis of the matrix coefficients of this set shows that it has the determinant of normal type and, hence, it can be solved by the reduction method. After we have determined the values of series coefficients in eqns (5) and (7) we can calculate the displacements, strains and stresses in any point of our structure model. The accuracy of solution is governed by the maximum value of index $t$ retained in eqns (6) and (7) or, the same, in the truncated set of equations in the numerical realisation of algorithm.

The most principal problem in the procedure outlined above is, of course, the representation of $\mathbf{u}^{(0)}$, defined by eqns (5) and (6), in a form, suitable for executing the interface contact conditions (4) on each separate particle. To perform the necessary transformation we need to expand the external solutions $\mathbf{S}_{ts}^{(i)}$ written in the $p$th local coordinate basis ($p = 1, 2, \ldots, N$) over the partial solutions, written in the $q$th local basis for $p \neq q$. The appropriate re-expansion formulae (addition theorems) for the case of two identically oriented coordinate systems are derived by Kushch (1995). They have for $\mathbf{r}_q = \mathbf{R}_{pq}+\mathbf{r}_q$ the following form:

$$\mathbf{S}_{ts}^{(i)}(\mathbf{r}_p, d_p) = \sum_{j=1}^{3}\sum_{k=0}^{\infty}\sum_{l=-k}^{k} \eta_{tksl}^{(i)(j)}(\mathbf{R}_{pq},f_p,f_q)\mathbf{s}_{kl}^{(j)}(\mathbf{r}_q,f_q) \tag{8}$$

for the explicit expressions of the coefficients $\eta_{tksl}^{(i)(j)}$ see (Kushch, 1996a).

It is fairly straightforward to show that the linear part of $\mathbf{u}^{(0)}$ can be represented by the series

$$\hat{E}\cdot\mathbf{r}_1 = \sum_{i=1}^{3}\sum_{t=0}^{\infty}\sum_{s=-t}^{t} b_{ts}^{(i)(q)}\mathbf{s}_{ts}^{(i)}(\mathbf{r}_q,f_q) \tag{9}$$

where

$$e_{10}^{(1)(q)} = E_{13}X_{1q}+E_{23}Y_{1q}E_{33}Z_{1q}$$

$$e_{11}^{(1)(q)} = -\overline{e_{q,-1}^{(1)(q)}} = (E_{11}-iE_{12})X_{1q}+(E_{12}-iE_{22})Y_{1q}+(E_{13}-iE_{23})Z_{1q}$$

$$e_{00}^{(3)(q)} = \frac{f_q}{2(2v_0-1)}(E_{11}+E_{22}+E_{33}),\quad e_{20}^{(1)(q)} = \frac{f_q}{3}(2E_{33}-E_{11}-E_{22})$$

$$e_{21}^{(1)(q)} = -\overline{e_{2,-1}^{(1)(q)}} = f_q(E_{13}-iE_{23}),\quad e_{22}^{(1)(q)} = \overline{e_{2,-2}^{(1)(q)}} = f_q(E_{11}-E_{22}-2iE_{12}) \tag{10}$$

all other $e_{ts}^{(i)(q)}$ are equal to zero. Values $X_{1q}$, $Y_{1q}$ and $Z_{1q}$ are the Cartesian coordinates of vector $\mathbf{R}_{1q}$ in a global coordinate basis.

By applying the formula (8) to the second term in eqn (5) we obtain after some algebra

$$\mathbf{u}^{(0)}(\mathbf{r}_q,f_q) = \sum_{i=1}^{3}\sum_{t=0}^{\infty}\sum_{s=-t}^{t}\left[A_{ts}^{(i)}\mathbf{S}_{ts}^{(i)}(\mathbf{r}_q,f_q)+\sum_{p=1}^{N}(a_{ts}^{(i)(p)(q)}+\delta_p^1 e_{ts}^{(i)(q)})\mathbf{s}_{ts}^{(i)}(O_{pq}\cdot\mathbf{r}_q,f_q)\right] \tag{11}$$

where

$$a_{ts}^{(i)(p)(q)} = \sum_{j=1}^{3}\sum_{k=0}^{\infty}\sum_{l=-k}^{k}\eta_{ktls}^{(j)(i)}(\mathbf{R}_{pq},f_p,f_q)A_{kl}^{(j)(p)}$$

$$a_{ts}^{(i)(q)(q)} = 0. \tag{12}$$

It is convenient to rewrite eqn (12) in a compact matrix mode

$$\mathbf{a}_t^{(p)(q)} = \sum_{k=0}^{\infty}\eta_{kt}^{(p)(q)}\cdot\mathbf{A}_k^{(p)} \tag{13}$$

where vector $\mathbf{A}_t^{(p)}$ contains the unknowns $A_{t+i-2,s}^{(i)(p)}$ and vector $\mathbf{a}_t^{(p)(q)}$ includes the values $a_{t-i+2,s}^{(i)(p)(q)}$. The structure of matrix $\eta_{kt}^{(p)(q)}$ is clear from eqns (12) and (13). It will be shown below that use of these matrix–vector notations simplifies greatly obtaining the resolving set of equations.

To complete the transformation of $\mathbf{u}^{(0)}$ we must re-expand the functions $\mathbf{s}_{ts}^{(i)}(O_{pq}\cdot\mathbf{r}_q,f_q)$ over the partial solutions written in the $q$th local spheroidal basis. To this end we use the formula (B12) derived in Appendix B. We rewrite now this formula as

$$\mathbf{s}_{ts}^{(i)}(O_{pq}\cdot\mathbf{r}_q,f_q) = \sum_{j=1}^{3}\sum_{k=0}^{t+i-j}\sum_{l=-k}^{k}Q_{tksl}^{(i)(j)}(\mathbf{w}_{pq},f_p,f_q)\mathbf{s}_{kl}^{(j)}(\mathbf{r}_q,f_q). \tag{14}$$

Its substitution into eqn (11) gives us the desired result

$$\mathbf{u}^{(0)} = \sum_{i=1}^{3}\sum_{t=0}^{\infty}\sum_{s=-t}^{t}[A_{ts}^{(i)(q)}\mathbf{S}_{ts}^{(i)}(\mathbf{r}_q,f_q)+q_{ts}^{(i)(q)}\mathbf{s}_{ts}^{(i)}(\mathbf{r}_q,f_q)] \tag{15}$$

where

$$q_{ts}^{(i)(q)} = \sum_{p=1}^{N}\sum_{j=1}^{3}\sum_{k=t+j-i}^{\infty}\sum_{l=-k}^{k}Q_{ktls}^{(j)(i)}(\mathbf{w}_{pq},f_p,f_q)(a_{ts}^{(i)(p)(q)}+\delta_p^1 e_{ts}^{(i)(q)}) \tag{16}$$

or, in a compact form

$$\mathbf{q}_t^{(q)} = \sum_{p=1}^{N} \sum_{k=t}^{\infty} Q_{kt}^{(p)(q)} \cdot (\mathbf{a}_k^{(p)(q)} + \delta_p^1 \mathbf{e}_k^{(q)}). \tag{17}$$

Vectors $\mathbf{q}_t^{(q)}$ and $\mathbf{e}_t^{(q)}$ have the same structure as $\mathbf{a}_t^{(q)}$, matrix $Q_{kt}^{(p)(q)}$ is determined uniquely by eqns (16) and (17). Now we recognize that $\mathbf{e}_t^{(q)} \neq 0$ for $t = 1$ only. Hence:

$$\mathbf{q}_t^{(q)} = Q_{11}^{(1)(q)} \cdot \delta_t^1 \mathbf{e}_t^{(q)} + \sum_{\substack{p=1\\p\neq q}}^{N} \sum_{k=t}^{\infty} Q_{kt}^{(p)(q)} \cdot \left( \sum_{n=0}^{\infty} \eta_{nk}^{(p)(q)} \cdot \mathbf{A}_n^{(p)} \right)$$

$$= Q_{11}^{(1)(q)} \cdot \delta_t^1 \mathbf{e}_t^{(q)} + \sum_{\substack{p=1\\p\neq q}}^{N} \sum_{n=0}^{\infty} \left( \sum_{k=t}^{\infty} Q_{kt}^{(p)(q)} \cdot \eta_{nk}^{(p)(q)} \right) \cdot \mathbf{A}_n^{(p)}. \tag{18}$$

After we have obtained the expression of $\mathbf{u}^{(0)}$ (15) the following course of solution is quite analogous to that described in detail by Kushch (1996a). Substitution of the expressions (15) and (7) into the contact conditions (4) and decomposition of the functional equalities obtained over a set of spherical harmonics $\chi_t^s(\eta_q, \phi_q)$ gives us the infinite set of linear algebraic equations

$$UG_t^{(q)}(v_0) \cdot \mathbf{A}_t^{(q)} + UM_t^{(q)}(v_0) \cdot \mathbf{q}_t^{(q)} = UM_t^{(q)}(v_q) \cdot \mathbf{D}_t^{(q)}$$

$$TG_t^{(q)}(v_0) \cdot \mathbf{A}_t^{(q)} + TM_t^{(q)}(v_0) \cdot \mathbf{q}_t^{(q)} = \omega_q TM_t^{(q)}(v_q) \cdot \mathbf{D}_t^{(q)}$$

$$q = 1, 2, \ldots, N; \quad t = 1, 2, \ldots; \tag{19}$$

where $\omega_q = \mu_q/\mu_0$, $\mu_q$ is the shear modulus of $q$th phase. Vector $\mathbf{D}_t^{(q)}$ contains the unknowns $D_{t-i+2,s}^{(i)(q)}$, the matrices $UM_t^{(q)}$, $UG_t^{(q)}$, $TM_t^{(q)}$ and $TG_t^{(q)}$ are defined in the paper cited. In the case of aligned spheroids we have $Q_{tt}^{(p)(q)} = I$ and $Q_{tk}^{(p)(q)} = 0$ for $t \neq k$. In this case the resolving set of equations (19) is reduced to that obtained before by Kushch (1996a). Note, finally, that the iterative solving procedure described there is also applicable in our case because the relations (14) are the exact finite identities and their application does not affect the properties of the matrix of infinite set (19).

## 4. NUMERICAL EXAMPLE

Numerical realization of the algorithm exposed above is rather simple. The most time consuming part of it is a calculation of the translation matrices $\eta_{tk}$, the rotation matrices $Q_{tk}$ and the correction matrices $UM_t$, $UG_t$, $TM_t$ and $TG_t$, containing an information on size, shape and properties of inhomogeneities. So, for a given type of inclusions we can calculate the correction matrices only once and use them repeatedly for the various configurations of particles. Again, when we consider the rotating particles with a fixed centre position there is no need to re-calculate each time the translation matrix which does not contain the orientation parameters of spheroids, etc. After we have calculated all these matrices for a given geometry of model, the rest of algorithm is merely the algebraic matrix–vector operations. Note, that by applying the iterative procedure to solve the algebraic set of equations we obtain the result without assembling the global matrix of system. Owing to these features the algorithm appears to be efficient from a computational standpoint and realisable even on the tiny models of personal computers.

An accuracy of the solution obtained by above method is determined entirely by the maximum value of index $t$ retained in the truncated set of equations (19) in a numerical realisation. Convergence of the approximate solution to the exact one with $t_{\max}$ increased as well as convergence of the iterative procedure in a case of aligned spheroids was studied by Kushch (1996a). The numerical experiments show that convergence of the solution with $t_{\max} \to \infty$ is rapid enough to exclude only the case of nearly-touching particles, characterized by a decrease of the convergence rate. The separate problem is the interacting nearly-touching rigid inclusions where the stresses tends to infinity as the particles draw together, i.e. this problem has a singularity point. This case requires a separate consideration by analogy with the asymptotic analysis carried out by Nunan and Keller (1984).

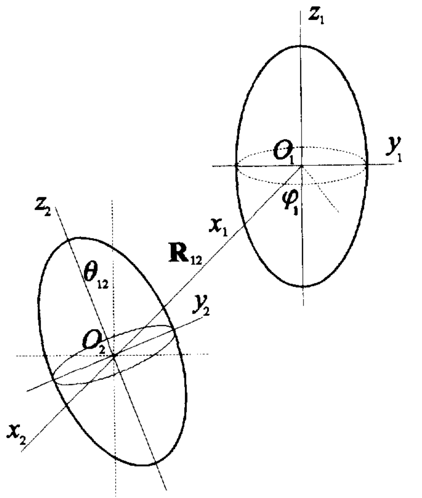


Fig. 1. Coordinates in the two-particle test problem.

The problem considered has a variety of parameters. They are the number of particles, their size, shape, position and orientation, the elastic moduli of matrix and inclusions and the external load tensor. The full parametric study of this problem is rather the subject of a separate paper. We present here only one numerical example which confirms a validity of the theory developed and illustrates how the orientation of the interacting spheroidal inhomogeneities may influence the stress field between them. To minimize a list of parameters we consider the simplest problem of this kind, namely, a space containing two equal spheroidal cavities ($\mu_1 = \mu_2 = 0$). The cavities are positioned so that their $Ox$-axes coincide (Fig. 1). The external load is the uniaxial deformation along $Ox$ axis $E_{11} = \delta$. The surface of cavities, as it follows from eqn (4), is stress-free. To concretize we put the Poisson's ratio of a matrix material $\nu_0 = 0.3$ and the aspect ratio of spheroid $\varepsilon = 2.0$. Under

Table 1. Convergence of the normalized stress $\sigma_{22}^{(0)}/\mu_0\delta$ on the first cavity equator with $t_{max}$ increased in the case of two aligned spheroidal cavities ($\theta_{12} = 0$)

| | $R_{12} = 2.1p_x$ | | $R_{12} = 2.2p_x$ | | $R_{12} = 2.5p_x$ | | $R_{12} = 3.0p_x$ | |
|---|---|---|---|---|---|---|---|---|
| $t_{max}$ | $\phi_1 = 0$ | $\phi_1 = \pi$ | $\phi_1 = 0$ | $\phi_1 = \pi$ | $\phi_1 = 0$ | $\phi_1 = \pi$ | $\phi_1 = 0$ | $\phi_1 = \pi$ |
| 1 | 1.83 | 1.83 | 1.77 | 1.77 | 1.62 | 1.62 | 1.43 | 1.43 |
| 3 | 3.98 | 1.65 | 3.61 | 1.56 | 2.94 | 1.41 | 2.28 | 1.27 |
| 5 | 4.95 | 1.35 | 4.29 | 1.27 | 3.37 | 1.21 | 2.56 | 1.16 |
| 7 | 5.25 | 1.29 | 4.38 | 1.22 | 3.39 | 1.17 | 2.60 | 1.13 |
| 9 | 5.47 | 1.28 | 4.42 | 1.23 | 3.37 | 1.17 | 2.60 | 1.13 |
| 11 | 5.67 | 1.27 | 4.46 | 1.23 | 3.36 | 1.18 | 2.59 | 1.13 |
| 13 | 5.80 | 1.27 | 4.48 | 1.23 | 3.35 | 1.18 | 2.59 | 1.13 |
| 15 | 5.83 | 1.27 | 4.49 | 1.23 | 3.35 | 1.18 | 2.59 | 1.13 |

Table 2. Convergence of the normalized stress $\sigma_{22}^{(0)}/\mu_0\delta$ on the first cavity equator with $t_{max}$ increased in the case of two cross-position spheroidal cavities ($\theta_{12} = \pi/2$)

| | $R_{12} = 2.1p_x$ | | $R_{12} = 2.2p_x$ | | $R_{12} = 2.5p_x$ | | $R_{12} = 3.0p_x$ | |
|---|---|---|---|---|---|---|---|---|
| $t_{max}$ | $\phi_1 = 0$ | $\phi_1 = \pi$ | $\phi_1 = 0$ | $\phi_1 = \pi$ | $\phi_1 = 0$ | $\phi_1 = \pi$ | $\phi_1 = 0$ | $\phi_1 = \pi$ |
| 1 | 1.46 | 1.46 | 1.44 | 1.44 | 1.38 | 1.38 | 1.29 | 1.29 |
| 3 | 2.83 | 1.41 | 2.60 | 1.35 | 2.20 | 1.41 | 1.86 | 1.12 |
| 5 | 3.17 | 1.17 | 2.81 | 1.13 | 2.37 | 1.21 | 2.03 | 1.11 |
| 7 | 3.34 | 1.17 | 2.82 | 1.13 | 2.32 | 1.17 | 2.04 | 1.10 |
| 9 | 3.36 | 1.21 | 2.76 | 1.17 | 2.23 | 1.17 | 2.02 | 1.10 |
| 11 | 3.45 | 1.20 | 2.74 | 1.16 | 2.20 | 1.18 | 2.01 | 1.10 |
| 13 | 3.54 | 1.20 | 3.73 | 1.17 | 2.20 | 1.18 | 2.01 | 1.10 |
| 15 | 3.57 | 1.20 | 2.73 | 1.17 | 2.20 | 1.18 | 2.01 | 1.10 |

these assumptions we have only two variable parameters in our model: they are the distance $R_{12}$ between the centres of spheroids and the angle $\theta_{12}$ characterizing the rotation of 2nd spheroid in $Oyz$-plane. The corresponding Euler's parameters are $w_1 = w_3 = 0$, $w_2 = -\sin[(1/2)\theta_{12}]$ and $w_4 = \cos[(1/2)\theta_{12}]$. Note, that even in this simplified to maximum possible extent statement we have an essentially three-dimensional problem.

The curves at the Figs 2–4 show distribution of the dimensionless stresses $\sigma_{ii}^{(0)}/\mu_0\delta$ on the equator of 1st cavity. The dotted curve 1 on each plot corresponds to value $R_{12} = \infty$ (single inclusion), in this case $\theta_{12}$ makes no effect on the stress fields around the cavity. The dashed lines 2 and 3 correspond to value $R_{12} = 2.5p_x$ and the solid lines 4 and 5 are calculated for $R_{12} = 2.1p_x$. In the latest case the cavities are placed closely enough: the distance between their surfaces is only 5% of distance between the centres of spheroids. The curves 2 and 4 represent the case of aligned spheroids $\theta_{12} = 0$ whereas the curves 3 and 5 are calculated for $\theta_{12} = \pi/2$ (cross-positioned spheroids). All calculations were performed with $t_{max} = 15$, providing a sufficient accuracy of calculations for all considered values of the parameters of model.

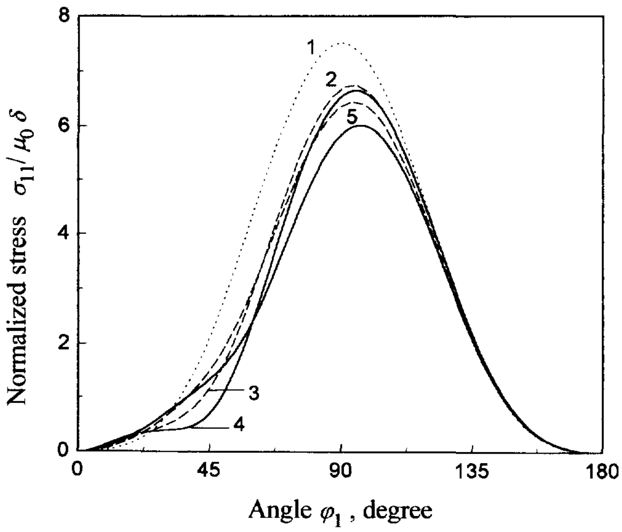


Fig. 2. Dimensional stress $\sigma_{11}^{(0)}/\mu_0\delta$ distribution of the equator of 1st cavity.

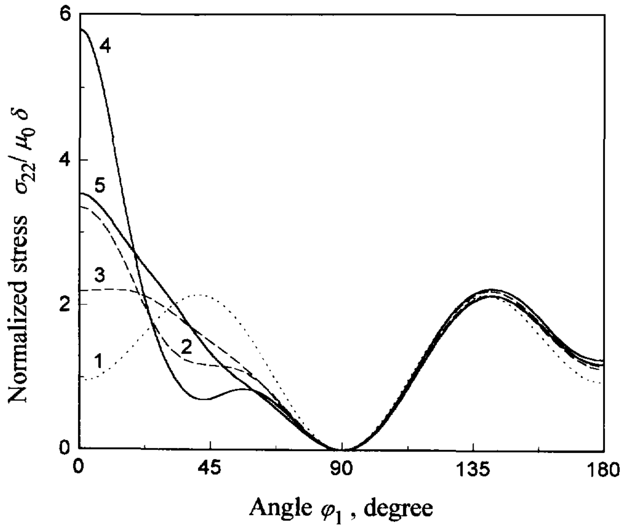


Fig. 3. Dimensional stress $\sigma_{22}^{(0)}/\mu_0\delta$ distribution of the equator of 1st cavity.

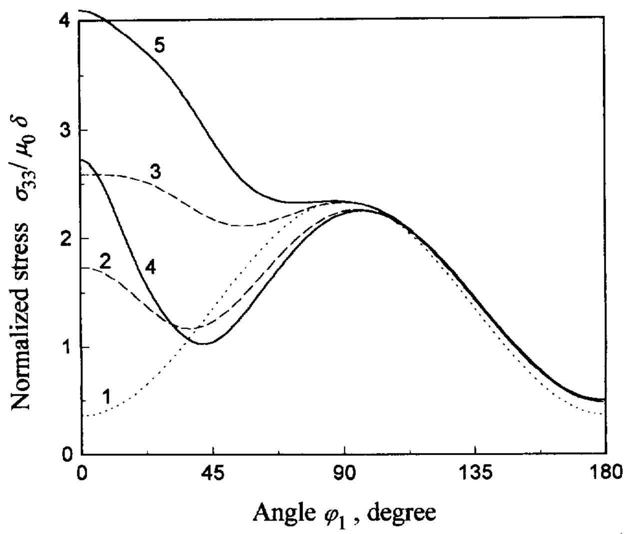


Fig. 4. Dimensional stress $\sigma_{33}^{(0)}/\mu_0\delta$ distribution of the equator of 1st cavity.

It is seen from the plots that the disturbance caused by another cavity is localized mainly at the front side of 1st spheroid ($0 \leqslant \phi_1 \leqslant \pi/2$), whereas on the back side $\pi/2 < \phi_1 \leqslant \pi$ the stresses are practically the same as for a single cavity. Note, that according to the boundary conditions $\sigma_{11}^{(0)} = 0$ at $\phi_1 = 0$. As a result, the stress $\sigma_{11}^{(0)}$ (Fig. 2) is only slightly depending on the parameters $R_{12}$ and $\theta_{12}$. On the contrary, the stresses $\sigma_{22}^{(0)}$ and $\sigma_{33}^{(0)}$ (Figs 3 and 5, respectively) are strongly influenced by these parameters. The smaller is the distance between the spheroids $R_{12}$ the greater is deviation from the solution for a single cavity. The effect of $\theta_{12}$ is not so unambiguous: so, $\sigma_{22}^{(0)}$ has a maximum at $\theta_{12} = 0$, whereas $\sigma_{33}^{(0)}$ peaks at $\theta_{12} = \pi/2$. Note, that for the oblate spheroids this dependence is inverse, for the spherical cavities $\sigma_{22}^{(0)} = \sigma_{33}^{(0)}$ and does not depend on $\theta_{12}$. Note, finally, that to consider the problem with a remote stress tensor $\hat{S}$ prescribed instead of $\hat{E}$ one must simply express the components of $\hat{S}$ through $\hat{E}$ and substitute the result into the expression of $\mathbf{e}_t$ (10).

## 5. CONCLUSIONS

The proposed rigorous analytical method to solve the elasticity theory problems in a multiple-connected domain with spheroidal interfaces is simple and effective from a computational standpoint. It generalized the approach developed before on the case of arbitrarily oriented spheroids and provides an accurate analysis of a variety of many-particle problems of composite mechanics. In particular, a periodic composite with an elementary structure cell containing a finite number of arbitrarily placed and oriented spheroidal particles can be analyzed analogously to that which was performed by Kushch (1996b) in a case of aligned spheroids. The serious advantage of presented solution is that it incorporates easily into a general scheme of so-called $O(N)$ algorithm [see e.g. Sangani and Mo, (1995)], providing an effective numerical study of interactions among the large number of particles. All the necessary for this purpose theoretical results are obtained by Kushch (1995).

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## APPENDIX A

*Vectorial partial solutions of Lame's equation in spheroidal coordinates*

The following vectorial functions, satisfying the Lame's equation written in spheroidal coordinates, have been entered by Kushch (1995):

(a) external, or singular, solutions $\mathbf{S}_{ts}^{(i)} = \mathbf{S}_{ts}^{(i)}(\mathbf{r},f)$, disappearing at $\|\mathbf{r}\| \to \infty$, are

$$\mathbf{S}_{ts}^{(1)} = \mathbf{e}_1 F_{t+1}^{s-1} - \mathbf{e}_2 F_{t+1}^{s+1} + \mathbf{e}_3 F_{t-1}^{s}$$

$$\mathbf{S}_{ts}^{(2)} = \frac{1}{t}[\mathbf{e}_1 (t+s) F_t^{s-1} + \mathbf{e}_2 (t-s) F_t^{s+1} + \mathbf{e}_3 s F_t^s]$$

$$\begin{aligned} \mathbf{S}_{ts}^{(3)} = \mathbf{e}_1 \{ & -(x-iy) D_2 F_{t-1}^{s-1} - [(\xi_0)^2 - 1] D_1 F_t^s + (t+s-1)(t+s)\beta_{-(t+1)} F_{t-1}^{s-1} \} \\ & + \mathbf{e}_2 \{ (x+iy) D_1 F_{t-1}^{s+1} - [(\xi_0)^2 - 1] D_2 F_t^s - (t-s-1)(t-s)\beta_{-(t+1)} F_{t-1}^{s+1} \} \\ & + \mathbf{e}_3 [z D_3 F_{t-1}^s - (\xi_0)^2 D_3 F_t^s - C_{-(t+1),s} F_{t-1}^s]. \end{aligned} \quad \text{(A1)}$$

(b) internal, or regular, solutions $\mathbf{s}_{ts}^{(i)} = \mathbf{S}_{ts}^{(i)}(\mathbf{r},f)$, constrained at $\mathbf{r} = 0$, are

$$\mathbf{s}_{ts}^{(1)} = \mathbf{e}_1 f_{t-1}^{s-1} - \mathbf{e}_2 f_{t-1}^{s+1} + \mathbf{e}_3 f_{t-1}^{s}$$

$$\mathbf{s}_{ts}^{(2)} = \frac{1}{(t+1)}[\mathbf{e}_1 (t-s+1) f_t^{s-1} + \mathbf{e}_2 (t+s+1) f_t^{s+1} - \mathbf{e}_3 s f_t^s]$$

$$\begin{aligned} \mathbf{s}_{ts}^{(3)} = \mathbf{e}_1 \{ & -(x-iy) D_2 f_{t+1}^{s-1} - [(\xi_0)^2 - 1] D_1 f_t^s + (t-s+1)(t-s+2)\beta_t f_{t+1}^{s-1} \} \\ & + \mathbf{e}_2 \{ (x+iy) D_1 f_{t+1}^{s+1} - [(\xi_0)^2 - 1] D_2 f_t^s - (t+s+1)(t+s+2)\beta_t f_{t-1}^{s+1} \} \\ & + \mathbf{e}_3 [z D_3 f_{t+1}^s - (\xi_0)^2 D_3 f_t^s - C_{ts} f_{t+1}^s] \end{aligned} \quad \text{(A2)}$$

where

$$\beta_t = \frac{t+5-4\nu}{(t+1)(2t+3)}, \quad C_{ts} = (t-s+1)(t+s+1)\beta_t, \quad t = 0, 1, \ldots, |s| \leqslant t.$$

In (A1)–(A2) the following notations are adopted:

$$\mathbf{e}_1 = (\mathbf{e}_x + i\mathbf{e}_y)/2, \quad \mathbf{e}_2 = (\mathbf{e}_x - i\mathbf{e}_y)/2, \quad \mathbf{e}_3 = \mathbf{e}_z$$
$$D_1 = (\partial/\partial x - i\partial/\partial y), \quad D_2 = \bar{D}_1 = (\partial/\partial x + i\partial/\partial y), \quad D_3 = \partial/\partial z. \tag{A3}$$

Scalar functions

$$f_t^s(r,f) = \frac{(t-s)!}{(t+s)!} P_t^s(\xi)\chi_t^s(\eta,\phi), \quad F_t^s(r,f) = \frac{(t-s)!}{(t+s)!} Q_t^s(\xi)\chi_t^s(\eta,\phi) \tag{A4}$$

are the internal and external partial solutions, respectively, of Laplace's equation in spheroidal coordinates $(f, \xi, \eta, \phi)$ defined as

$$x + iy = f\bar{\xi}\bar{\eta}\exp(i\phi), \quad z = f\xi\eta$$
$$\bar{\xi}^2 = \xi^2 - 1, \quad \bar{\eta}^2 = 1 - \eta^2, \quad 1 < \xi < \infty, \quad 0 \leqslant \theta \leqslant \pi, \quad |\eta| \leqslant 1, \quad 0 \leqslant \phi \leqslant 2\pi. \tag{A5}$$

In eqn (A4) $\chi_t^s(\eta, \phi) = P_t^s(\eta)\exp(i\phi)$ are the scalar spherical harmonics, $P_t^s$ and $Q_t^s$ are the associated Legendre's polynomials of first and second kind, respectively. The equalities (A5) at $Re(f) > 0$ describe the family of confocal prolate spheroids with inter-foci distance $2f$. The functions (A1) and (A2) are written here in a prolate-spheroidal basis. In the case of an oblate spheroid one must replace $\xi$ on $i\bar{\xi}$ and $f$ on $(-if)$ in (A1) and all the following formulae.

## APPENDIX B

*Re-expansion formulae for the internal solutions of Lame's equation due to rotation of a coordinate system*

We begin with the classical formula by Bateman and Erdelyi (1953) describing a transformation of the scalar spherical harmonics $\chi_t^s(\mathbf{r}) = P_t^s(\cos\theta)\exp(is\phi)$ due to rotation of a coordinate basis:

$$\chi_t^s(O\cdot\mathbf{r}) = \sum_{l=-t}^{t} \frac{(t-l)!}{(t-s)!} S_{2t}^{t-s,t-l}(\mathbf{w})\chi_t^l(\mathbf{r}) \tag{B1}$$

where $S_t^{s,l}$ are the spherical harmonics in a four-dimensional space and $\mathbf{w} = \{w_1, w_2, w_3, w_4\}$ is the vector of Euler's parameters, determining uniquely the rotation matrix

$$O = \begin{pmatrix} w_2^2 - w_1^2 - w_3^2 + w_4^2 & 2(w_2w_3 - w_1w_4) & 2(w_1w_2 + w_3w_4) \\ 2(w_2w_3 + w_1w_4) & w_3^2 - w_1^2 - w_2^2 + w_4^2 & 2(w_1w_3 - w_2w_4) \\ 2(w_1w_2 - w_3w_4) & 2(w_1w_3 + w_2w_4) & w_1^2 - w_2^2 - w_3^2 + w_4^2 \end{pmatrix} \quad \|\mathbf{w}\| = w_1^2 + w_2^2 + w_3^2 + w_4^2 = 1. \tag{B2}$$

The formulae, analogous to eqn (B1), can be obtained also for the vectorial spherical harmonics (Morse and Feshbach, 1953), written in the following form:

$$\mathbf{P}_{ts}^{(1)} = \mathbf{e}_r\chi_t^s(\theta,\phi)$$
$$\mathbf{P}_{ts}^{(2)} = \mathbf{e}_\theta\frac{\partial}{\partial\theta}\chi_t^s(\theta,\phi) + \frac{\mathbf{e}_\phi}{\sin\theta}\frac{\partial}{\partial\phi}\chi_t^s(\theta,\phi)$$
$$\mathbf{P}_{ts}^{(3)} = \frac{\mathbf{e}_\theta}{\sin\theta}\frac{\partial}{\partial\phi}\chi_t^s(\theta,\phi) - \mathbf{e}_\phi\frac{\partial}{\partial\theta}\chi_t^s(\theta,\phi). \tag{B3}$$

Taking into account the expression of $\mathbf{P}_{ts}^{(1)}$ and the invariant differential relations $r\nabla\times\mathbf{P}_{ts}^{(2)} = -\mathbf{P}_{ts}^{(3)}$ and $r\nabla\times\mathbf{P}_{ts}^{(1)} = \mathbf{P}_{ts}^{(3)}$ (Golovchan, 1974), we obtain easily that

$$\mathbf{P}_{ts}^{(i)}(O\cdot\mathbf{r}) = \sum_{l=-t}^{t} \frac{(t-l)!}{(t-s)!} S_{2t}^{t-s,t-l}(\mathbf{w})\mathbf{P}_{tl}^{(i)}(\mathbf{r}), \quad i = 1,2,3. \tag{B4}$$

The transformation formulae for the internal partial solutions of Lame's equation in spherical coordinates $(r, \theta, \phi)$ (see, e.g. Golovchan *et al.*, 1993)

$$\mathbf{u}_{ts}^{(1)} = \frac{r^{t-1}}{(t+s)!}(t\mathbf{P}_{ts}^{(1)} + \mathbf{P}_{ts}^{(2)}); \quad \mathbf{u}_{ts}^{(2)} = \frac{r^t}{(t+s)!}\mathbf{P}_{ts}^{(3)}$$
$$\mathbf{u}_{ts}^{(3)} = \frac{r^{t+1}}{(t+s)!}(\gamma_t\mathbf{P}_{ts}^{(1)} + \beta_t\mathbf{P}_{ts}^{(2)}) \tag{B5}$$

where

$$\gamma_t = \frac{t-2+4v}{2t+3} \quad \text{and} \quad \beta_t = \frac{t+5-4v}{(t+1)(2t+3)}$$

follow directly from eqn (B4). Because $r$ is invariant on an arbitrary rotating of the coordinate system with a fixed origin, we find the simple formula

$$\mathbf{u}_{ts}^{(i)}(O \cdot \mathbf{r}) = \sum_{l=-t}^{t} \frac{(t-l)!(t+l)!}{(t-s)!(t+s)!} S_{2t}^{l-s,t-l}(\mathbf{w}) \mathbf{u}_{tl}^{(i)}(\mathbf{r}). \tag{B6}$$

In this and all the following analogous formulae we keep in mind that the vectorial functions standing in the opposite sides of equality are written in their own local basis.

Now we make use of the following two identities, that relate the internal partial solutions of Lame's equation in spherical and spheroidal coordinates. The first formula has a form ($\xi^{(0)} = 0$)

$$\mathbf{s}_{ts}^{(i)}(\mathbf{r},f) = \sum_{j=1}^{i} \sum_{k=|s|+j-1}^{t} {}^{(1)}K_{tks}^{(i)(j)}(f) \mathbf{u}_{k+i-j,s}^{(j)}(\mathbf{r}) \tag{B7}$$

where

$${}^{(1)}K_{tks}^{(i)(j)} = 0 \quad \text{for } i < j; \quad {}^{(1)}K_{tks}^{(1)(1)} = K_{t-1,k-1}^{(1)}, \quad {}^{(1)}K_{tks}^{(2)(2)} = K_{tk}^{(1)}$$

$${}^{(1)}K_{tks}^{(3)(3)} = K_{t+1,k+1}^{(1)}, \quad {}^{(1)}K_{tks}^{(2)(1)} = \left(\frac{s}{k+1} - \frac{s}{t+1}\right) K_{tk}^{(1)}$$

$${}^{(1)}K_{tks}^{(3)(2)} = 4(1-v)\left(\frac{s}{k+1} - \frac{s}{t+1}\right) K_{t+1,k+1}^{(1)}$$

$${}^{(1)}K_{tks}^{(3)(1)} = \left[C_{-(t+3),s} - C_{-(k+3),s} + 4(1-v)\frac{s}{(t+1)}\left(\frac{s}{k+2} - \frac{s}{t+2}\right)\right] K_{t+1,k+1}^{(1)}.$$

$$K_{tk}^{(1)}(f) = \frac{(-1)^{(t-k)/2}}{\sqrt{\pi}} \left(\frac{2}{f}\right)^k \frac{\Gamma\left(\frac{t+k}{2} + \frac{1}{2}\right)}{\left(\frac{t-k}{2}\right)!} \quad \text{for } t-k \text{ even}$$

$$K_{tk}^{(1)}(f) = 0 \quad \text{for } t-k \text{ odd}. \tag{B8}$$

The inverse relations are

$$\mathbf{u}_{ts}^{(i)}(\mathbf{r}) = \sum_{j=1}^{i} \sum_{k=|s|-j-i}^{t} {}^{(2)}K_{tks}^{(i)(j)}(f) \mathbf{s}_{k+i-j,s}^{(j)}(\mathbf{r},f) \tag{B9}$$

where the coefficients ${}^{(2)}K_{tks}^{(i)(j)}$ have a form (B8) with the replace $K_{tk}^{(1)}(f)$ on

$$K_{tk}^{(2)}(f) = \sqrt{\pi}\left(\frac{f}{2}\right)^t \frac{(k+1/2)}{\left(\frac{t-k}{2}\right)!\,\Gamma\left(\frac{t+k}{2} + \frac{3}{2}\right)} \quad \text{for } t-k \text{ even}$$

$$K_{tk}^{(2)}(f) = 0 \quad \text{for } t-k \text{ odd}. \tag{B10}$$

The formulae (B6), (B7) and (B9) have a certain practical value in themselves. Here we use them as the auxiliary formulae to express $\mathbf{s}_{ts}^{(i)}(\mathbf{r}_2,f_2)$ through $\mathbf{s}_{kl}^{(j)}(\mathbf{r}_1,f_1)$, where $\mathbf{r}_2 = O \cdot \mathbf{r}_1$ and, in a general case, $f_1 \neq f_2$. To complete this transformation we substitute eqn (B6) into right-hand side of eqn (B7) and, then, expand the expression obtained with aid of the relations (B9). After a change of summation order we obtain the desired finite series expansion:

$$\mathbf{s}_{ts}^{(i)}(\mathbf{r}_2,f_2) = \sum_{j=1}^{3} \sum_{k=0}^{t+i} \sum_{l=-k}^{k} Q_{tksl}^{(i)(j)}(\mathbf{w}_{12},f_1,f_2) \mathbf{s}_{kl}^{(j)}(\mathbf{r}_1,f_1) \quad i = 1,2,3; \quad t = 0,1,2,\ldots; \quad |s| \leqslant t \tag{B11}$$

where

$$Q_{tksl}^{(i)(j)} = \sum_{\alpha=j}^{i} \sum_{p=k-j-\alpha}^{t+i-\alpha} {}^{(1)}K_{t,p+\alpha-i,s}^{(i)(\alpha)}(f_2) {}^{(2)}K_{p,k+j-\alpha,l}^{(\alpha)(j)}(f_1)$$

$$\times \frac{(p-l)!(p+l)!}{(p-s)!(p+s)!} S_{2p}^{l-s,p-l}(\mathbf{w}_{12}) + \delta_i^3 \delta_j^1 (\xi^{(0)})^2 (2k-1) \sum_{n=k}^{t} (Q_{nsl}^{(3)(3)} - Q_{nsl}^{(2)(2)}) \tag{B12}$$

and $\delta_i^j$ is the Kronecker's symbol.